Extended Cobordism Hypothesis

A survey of an overview

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$Cob_1(n)$

$Cob_1(n)$ is the symmetric monoidal category whose:

- Obs: n-1 dim closed manifolds M
- Mors: n dim Bordisms B:M→N
- Comp: $B_1 \circ B_0 = B_0 \coprod_N B_1$
- $\bigotimes = \coprod$, $1_\otimes = \emptyset$

A dual of an object $X \in \mathfrak{C}$ is an object $\bar{X}$ with morphisms:

$$ev_X : \bar{X} \otimes X \to 1 \text{ and } coev_X : 1 \to X \otimes \bar{X}$$

Duals

In $Cob_1(n)$ we have the duals:

$$ev_M : \bar{M} \coprod M \to \emptyset \text{ and } coev_M : \emptyset \to M \otimes \bar{M}$$

[macaroni picture]

In $(Vect_k, \otimes, 1 = k)$ we have $ev_V : \bar{V} \otimes V \to k$ by

$$ev(f, v) = f(v)$$

when V is finite dimensional we have $coev_V : k \to V \otimes \bar{V}$ by

$$coev(k) = \sum^n kv_i \otimes v^i$$

Fully Dualizable

We say that $\mathfrak{C}$ is fully dualizable when all of it's objects have duals:

- $Cob_1(n)$ is fully dualizable
- $Vect_k$ is not, but $Vect_k^{fd}$ is

Let $\mathfrak{D}$ be fully dualizable and $F \in Fun^\otimes(\mathfrak{D}, \mathfrak{C})$, then F factors uniquely as

$$\mathfrak{D} \to \mathfrak{C}^{fd} \to \mathfrak{C}$$

Classical TQFTs

A TQFT is a SMFunctor

$$Z : Cob_1(n) \to Vect_k$$

by the above theorem, this reduces to

$$Z : Cob_1(n) \to Vect_k^{fd}$$

The fact that the essential image of Z must lie in the fully dualizable objects will allow us to reduce the calculation of Z.

Reduction Arguement

- Let $B : M \to N$ be a bordism
- rewrite as $B : \emptyset \to N \coprod \bar{M}$
- That is $B : \emptyset \to \partial B$ hence $Z(B) : k \to Z(\partial B)$
- Factor B as $M \overset{B_0}{\to} L \overset{B_0}{\to} N$
- $\partial(B_0 \coprod B_1) \cong \partial(B) \coprod \bar{L} \coprod L$

We have the commutative diagram:

$$\begin{array}{ccc} k & \xrightarrow{Z(B)} & Z(\partial B) \\ \downarrow{\scriptstyle Z(B_0)\otimes Z(B_1)} & & \uparrow{\scriptstyle Z(ev_L)} \\ Z(\partial B_0) \otimes Z(\partial B_1) & \xrightarrow{\sim} & Z(\partial B) \otimes Z(L) \otimes Z(L)^\wedge \end{array}$$

Allowing us to express $Z(B)$ in terms of $Z(B_0), Z(B_1)$ and $Z(ev_L)$

Classical Cobordism Hypotheses

Coupled with a classification of 0,1,2 manifolds, we get:

$$ev_* : Fun^{\otimes}(Cob_1(1), Vect_k^{fd}) \cong Vect_k^{fd} \text{ (as sets)}$$

A 1-dimensional TQFT is classified by Z(*)

and

$$Fun^{\otimes}(Cob_1(2), Vect_k^{fd}) \cong k - FrbAlg$$

A 2-dim TQFT is classified by Z on:

S^1, a "pair of pants" $S^1 \coprod S^1 \to S^1$ and a "cap" $S^1 \to \emptyset$

Reasons to Extend to (∞, n)-categories

Cob(n) should be an n-category:

- Classification of n>>2 manifolds is hard
- Covariance: $Cob_1(n)$ "Chooses" a time axis

Cob(n) should be an (inf, n)-category

- We want to explicitly deal with diffeomorphism
- We want to consider TQFT variants such as
  $$Z : Cob(n) \to \mathfrak{Chain}_k$$

i.e. we want to consider homotopy as well

Segal Spaces

Recall: a simplicial set X is the nerve of a category iff the diagram

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array} \quad \text{is a pullback}$$

Likewise a **Simplicial Space** $X : \Delta^{op} \to \mathfrak{Top}$ is a Segal Space if the above diagram is a **homotopy pullback** $X_{m+n} \simeq X_m \times^R_{X_0} X_n$

(inf, 1)-categories

for $x, y \in X_0$ let $Map(x, y) = \{x\} \times^R_{X_0} X_1 \times^R_{X_0} \{y\}$

define the homotopy category hX:
- Obs = X_0
- Mors = $\pi_0(Map(x, y))$

$f \in X_1$ is invertible if $[f] \in hX$ is. Let $Z \subseteq X_1$ be the invertibles.

$\delta_0 : X_0 \to X_1$ factors through $\delta_0 : X_0 \to Z$.
We say X is complete if δ_0 is a weak equivalence.

An $(\infty, 1)$-category is a complete segal space X

(∞, n)-categories

An n-fold simplicial space is a functor $X : (\Delta^{op})^n \to \mathfrak{Top}$.
Let $n\mathfrak{Top} = \{X : (\Delta^{op})^n \to \mathfrak{Top}\}$.
by **currying** we get $n\mathfrak{Top} = \{X : \Delta^{op} \to (n-1)\mathfrak{Top}\}$

an n-fold simplicial space is a (∞, n)-category if:
1) X satisfies the segal condition
2) X_0 is essentially constant
3) X_k are $(\infty, n-1)$-categories
4) $Y_k = X_{k,0,..,0}$ is an $(\infty, 1) - category$

$Bord_n$ as $(\infty, 1)$-category

Let $PreCob^V_k(n) = \{(M \hookrightarrow V \times \mathbb{R}, (a_0 \leq ... \leq a_k)\}$
(with some extra conditions)

[picture of transverse embedding]

$PreCob^V_.(n)$ has a natural topology making it a segal space, we can complete it to an $(\infty, 1)$-category $Bord_1(n)$:

- Obs: n-1 closed manifolds
- Mors: n Bordisms
- 2-Mors: Diffeomorphisms of Bordisms
- 3-Mors: isotopies of Diffeomorphisms

...

$Bord_n$

Let $PreCob^V_{k_1,\ldots,k_r}(n) = \{(M \hookrightarrow V \times \mathbb{R}^r, (a^i_0 \leq \ldots \leq a^i_{k_i})_{i \leq r}\}$
(with some extra conditions)

[picture of multi-transverse embedding]

$PreCob^V_\bullet(n)$ has a natural topology making it an n-fold segal space, we can complete it to an (∞, n)-category $Bord_n$:

- Obs: Points
- Mors: 1- Bordisms

...

- n-Mors: n-Bordisms
- (n+1)-Mors: Diffeomorphisms of Bordisms
- (n+1)-Mors: Isotopies

...

Adjoints

Let $\mathfrak{C}$ be a 2-category. We say $f \dashv g$ if

$$\eta : Id_X \to g \circ f \text{ and } \epsilon : f \circ g \to Id_Y$$
(Acting like the units of an adjunction)

$\mathfrak{C}$ has adjoints if all f has $g_l \dashv f \dashv g_r$

Let X be an $(\infty, n)-$category $n \geq 2$, define h_2X:
-Obs = X_0
-Mors = $X_{1,0}$
- 2-Mors = $\pi_0(Map(f, g))$

X has adjoints for 1 morphisms if h_2X has adjoints X

has adjoints for k morphisms if $Map(x, y)$ has adjoints for k-1 morphisms

$Bord_n$ is fully dualizable

We say a symmetric monoidal (∞, n)-category is fully dualizable if it has duals, and adjoints for all k

$Bord_n$ is fully dualizable
Thus $Fun^\otimes(Bord_n, \mathfrak{C}) \simeq Fun^\otimes(Bord_n, \mathfrak{C}^{fd})$

Moreover, the duals realize $Fun^\otimes(Bord_n, \mathfrak{C})$ as an ∞-groupoid:

$$\alpha : Z \to Z'$$

$$\alpha_M : Z(M) \to Z'(M) \text{ and } \alpha_{\bar{M}} : Z(\bar{M}) \to Z'(\bar{M})$$

$$\bar{\alpha}_{\bar{M}} : Z'(M) \to Z'(M)$$

$$\text{with } \bar{\alpha}_{\bar{M}} = \alpha_M^{-1}$$

(fake) Cobordism Hypothesis

This leads us to the (fake) cobordism hypothesis:

$$ev_* : Fun^\otimes(Bord_n, \mathfrak{C}) \to \mathfrak{C}^\sim$$

where $\mathfrak{C}^\sim$ is the core ∞-groupoid of a $(\infty, n)-$category

Framings

The real cobordism hypothesis is a lot weaker:

$$ev_* : Fun^{\otimes}(Bord_n^{fr}, \mathfrak{C}) \to \mathfrak{C}^{\sim}$$

It requires that we have n-framings on the manifolds in $Bord_n$.

Let M^m be a manifold, an n-framing on M is an isomorphism of vector bundles

$$T_M \oplus \mathbb{R}^{n-m} \cong \mathbb{R}^n$$

Let $Bord_n^{fr}$ be the $(\infty, n) - category$ of bordisms with n-framings

O(n)-action

This version of the cobordism hypothesis is not very interesting, namely it classifies TQFTs on manifolds with n-framings: **barely any manifolds have n-framings**.

What's interesting is that it induces an O(n)-action on $\mathfrak{C}^{\sim}$:
- O(n) acts on n-framings of a manifold M
- O(n) acts on $Bord_n^{fr}$
- O(n) acts on $Fun^{\otimes}(Bord_n^{fr}, \mathfrak{C}) \simeq \mathfrak{C}^{\sim}$

(X, ζ)-Structures

an n-framing is a specific case of an (X, ζ)-Structure.

Let X be a topological space, with a rank n vector bundle ζ with an inner product (v,w). An (X, ζ)-structure on a manifold M^m is:

- A map $f : M \to X$
- An isomorphism of vector bundles

$$T_M \oplus \mathbb{R}^{n-m} \cong f^*\zeta$$

Let $Bord_n^{(X,\zeta)}$ be the (∞, n)-category of Bordisms with a (X, ζ)-structure

The (real) Cobordism Hypothesis

For $x \in X$, let $F_x = \{R^n \xrightarrow{\sim} \zeta_x \text{ orthonormal}\}$. F_x carries an O(n) action by pre-composition, Let $\tilde{X} = \coprod_x F_x$ be the associated principal O(n)-bundle of frames in X.

We then have the following equivalence:

$$Fun^{\otimes}(Bord_n^{(X,\zeta)}, \mathfrak{C}) \simeq Hom_{O(n)}(\tilde{X}, \mathfrak{C}^{\sim})$$

Which is induced as follows:
Each $\tilde{x} \in \tilde{X}$ gives an (X, ζ)-structure on the point * (by pullback).
The restriction functor $res_*^{(X,\zeta)}$ induces the above equivalence

Sketch of the proof: Constructions

- Let (X, ζ)
- Write $\tilde{X} = \{(x, f) : f : \mathbb{R}^n \to \zeta_x\}$
- Let $X_0 = \{(x, v) | v \in \zeta_x (v, v) = 1\}$ the **sphere bundle**
- Let $\zeta_O = \{(x, v, w) | (x, v) \in X_0, (v, w) = 0\}$
- Let $p : X_0 \to X$ the projection, then

$$\zeta_0 \oplus \underline{\mathbb{R}} \cong p^* \zeta$$

with this equivalence any dim < n manifold with an (X_0, ζ_0)-structure also carries an (X, ζ)-structure. Thus:

$$Bord_{n-1}^{(X_0, \zeta_0)} \hookrightarrow Bord_n^{(X, \zeta)}$$

Sketch of the proof: Extension

- Let $\Omega X = Map_X(*, *)$, inductively define $\Omega^k X$
- For $x \in X$ let $S^{\zeta_x} = \{|v| = 1\} \in \Omega^{n-1} Bord_{n-1}^{(X_0, \zeta_0)}$
- For $Z : Bord_n^{(X, \zeta)} \to \mathfrak{C}$, restrict to $Z_0 : Bord_{n-1}^{(X_0, \zeta_0)} \to \mathfrak{C}$
- define $\Phi : X \to \Omega^{n-1} \mathfrak{C}$ by $\Phi(x) = Z_0(S^{\zeta_x})$

for a point $(x, v) \in X_0$ we can decompose S^{ζ_x} as

$$\emptyset \overset{S^{\zeta_x}_{v,+}}{\to} S^{\zeta_x}_{v,0} \overset{S^{\zeta_x}_{v,-}}{\to} \emptyset$$

yeilding a composite of morphisms in $\Omega^{n-1} \mathfrak{C}$

$$1 \overset{H_-(x,v)}{\to} H_0(x, v) \overset{H_+(x,v)}{\to} 1$$

non-degenerate morphisms

-Let $D^{\zeta_x} = \{|v| \leq 1\}$, $D^{\zeta_x} : \emptyset \to S^{\zeta_x}$

$$\eta_x = Z(D^{\zeta_x}) : 1 \to H_+(x, v) \circ H_-(x, v)$$

η_x witnesses $H_-(x, v) \dashv H_+(x, v)$. we say that η_x is non-degerate

Theorem 3.1.8

Let $Z_0 : Bord_{n-1}^{(X_0, \zeta_0)} \to \mathfrak{C}$, TFAE:

- $Z : Bord_{n-1}^{(X, \zeta)} \to \mathfrak{C}$ extending Z_0
- families of non-degenerate n-morphisms $\{\eta_x : 1 \to Z_0(S^{\zeta_x})\}_X$

Inductive Idea

Letting $(X, \zeta) = (*, \mathbb{R}^n)$

$$Bord_n^{fr} = Bord_n^{(*, \mathbb{R}^n)}$$

- Assume the General Cobordism Hypothesis for dimension n-1
- Let $Z : Bord_n^{fr} \to \mathfrak{C}$, Theorem 3.1.8 allows us to reduce to $Z_0 : Bord_{n-1}^{(S^{n-1}, \zeta_0)} \to \mathfrak{C}$ and a family $\eta_x : 1 \to Z_0(S^{n-1})$
- It follows from some highly technical details (p. 54-57) that this reduces to a functor $Z_- : Bord_{n-1}^{fr} \to \mathfrak{C}$
- This proves the Framed Cobordism Hypothesis in dimension n

General Cobordism Hypothesis

We will extend to the general cobordism hypothesis as follows:
Let (X, ζ) as before, consider $f : Y \to X \in \mathfrak{Top}/X$, we have two functors:

$$F(f) = Fun^{\otimes}(Bord_n^{(Y,f^*\zeta)}, \mathfrak{C}) \text{ and}$$
$$G(f) = Map_{O(n)}(\tilde{X} \times_X Y, \mathfrak{C})$$

restriction to the $(Y, f^*\zeta)$ structures on * gives us a natural

$$\alpha_f : F(f) \to G(f)$$

- α_f is an equivalence when $f : * \to X$
- Both F, G send homotopy colimits to homotopy limits
- all CW-complexes are generated by homotopy colimits of *
- α_f is always an equivalence, in particular α_{Id_X}